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Heterogeneity and Option Pricing

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Abstract. An economy with agents having constant yet heterogeneous degrees of relative risk aversion prices assets as though there were a single decreasing relative risk aversion "pricing representative" agent. The pricing kernel has fat tails, and option prices do not conform to the Black-Scholes formula. Implied volatility exhibits a "smile." Heterogeneity as the source of non-stationary pricing fits Rubenstein's (1994) interpretation of the "over-pricing" as an indication of "crash-o-phobia". Rubinstein's term suggests that those who hold out-of-the money put options have relatively high risk aversion (or relatively high subjective probability assessments of low market outcomes). The essence of this explanation is investor heterogeneity.

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I. Introduction

In this paper we investigate the pricing of assets in an economy in which there are multiple agents with heterogeneous tastes. The Arrow-Debreu model of general equilibrium under uncertainty does not restrict the heterogeneity of either the probability beliefs or the preferences of investors. In contrast, in the theories of asset pricing that followed Lucas (1978) there typically exists a representative consumer-investor whose preferences and probability assessments of the economy's stochastic endowment price all assets. This representative investor is almost always assumed to have time additive preferences with constant relative risk aversion. It is now well-known that the equilibrium framework necessary to derive the Black-Scholes formula for options, given a proportional Brownian diffusion of the underlying payout, requires the existence of such a representative consumer with constant relative risk aversion (see Rubinstein (1976), Breeden and Litzenberger (1978), Brennan (1979), Stapleton and Subrahmanyam (1984), Bick (1987, 1990), and He and Leland (1993)).

The assumption that all investors have identical homothetic tastes and identical expectations seems particularly unreasonable.[1] It is well known that this assumption implies that all investors have identical wealth composition. The empirical evidence seems to contradict this assumption: Mankiw and Zeldes (1991), for example, report that families that do not own any stock account for 62% of disposable income. Another recent study finds that in 1989 the top one percent of wealth holders held 36.2% of the total non-human worth of United States households and 62.5% of the business assets and corporate stock held by households (Kennickell and Woodburn, 1992). In addition, while a representative-agent framework may price all assets, it does not explain why there exists open interest in assets with zero net supply, such as the options, with investors on both sides (short or long) of the market. Some heterogeneity among investors, in either endowments, tastes or opinions seems necessary to explain why such assets will exist at all.[2]

The formal analysis of the equilibrium underpinnings of the Black-Scholes option pricing formula (for example Bick (1987, 1990) and He and Leland (1993)) retains the presumption of a representative agent. Yet *heterogeneity* among investors is embedded in most informal discussions of options markets. For example, Cox and Rubinstein (1985, p. 54) give the "use of certain kinds of special knowledge" as one reason for the existence of trade in options. According to such popular views, agents with bullish expectations (perhaps based on "special knowledge") will be attracted to out-of-the-money calls (written, presumably, by others with more bearish expectations). On the other hand, out-of-the-money put options are considered to be bought by bearish investors or by investors who are particularly concerned about down-side risk.[3]

We believe that heterogeneity among agents may also be the key for resolving the empirical non-congruence of the Black-Scholes formula which has attracted a sizeable literature in recent years. In an essentially a-theoretical framework, Rubinstein (1994) and others have attempted to derive a pattern of Arrow-Debreu pricing that is implied by observed option prices (and is inconsistent with the Black-Scholes framework). Franke et al. (2000) attempt to reconcile this observed pattern within an equilibrium framework where the representative agent displays declining relative risk aversion. In a related paper, Mathur and Ritchken (1999) also examine this issue. They show that under some conditions on aggregate output, when the representative has decreasing relative risk aversion, the Black-Scholes price will be the minimum market price for an option. Neither of these papers examines the role of heterogeneity in risk aversion scrutinized in this paper, in which we examine the case of the pricing of assets and options when all agents have the standard constant-elasticity tastes, but when agents' tastes differ.

One may wonder however whether consumer heterogeneity *per se* could matter. The early formulators of the CAPM were concerned about the effects of the assumption of homogeneity of opinion. As Sharpe (1970) wrote (p. 104): "Even the most casual empiricism suggests that this [homogeneous opinions] is not the case. People often hold passionately to beliefs that are far from universal." His conclusion, however, was that heterogeneity of opinion is by and large irrelevant since (p. 291) " in a somewhat superficial sense, the equilibrium relationships derived for a world of complete agreement can be said to apply to a world in which there is disagreement, if certain values are considered to be averages." In a similar vein, Constantinides (1982) established that the asset prices that arise in an economy with heterogeneous agents could be rationalized as if originating from the preferences of a single pricing-representative agent.[4]

In Mayshar (1983), one of us argued for the pricing relevance of heterogeneity of opinion, claiming that when some investors are in a corner solution (e.g. all those potential investors in the world who do not hold any particular asset), the sources of the heterogeneity that explain the corner solutions are relevant for the determination of what are the relevant "averages" and thus also for the pricing of assets. Corner solutions, and the identity of actual versus potential investors, are relevant also for the case of heterogeneity in tastes.

In this paper we advance a different argument against the practice of assuming a "representative" investor. As is common in the related literature, we assume that markets are Arrow-Debreu complete, and that all the heterogeneous consumer-investors have “reasonable” time-separable, constant elasticity utility functions with constant time-discount factors, so that no corner solutions exist. We demonstrate that within this framework, when consumers differ with respect to their risk aversion, the induced preferences of Constantinides’s pricing-representative agent are considerably more complicated than those of the actual agents in the economy, and in particular do not belong to the same class of “reasonable” preferences as theirs. In particular, we show that when consumers have different *constant* risk aversions, the pricing-representative agent’s preferences exhibit *declining* relative risk aversion. The pricing representative consumer’s preferences are thus not of the same class as those of the consumers he “represents.” We demonstrate the significance of this result, and of other sources of investor heterogeneity, for the pricing of options. We show that investor heterogeneity provides a simple and intuitive explanation for the empirical puzzle concerning the non-congruity of the Black-Scholes formula for option pricing with reality. These results cast considerable doubt on the standard practice in the literature of endowing the “representative” agent with “reasonable” (i.e., constant relative risk aversion) preferences.

The structure of the paper is as follows: In the following section we set out the equilibrium model, in which all heterogeneous consumers live for two dates. In Section III we characterize the preferences of the “pricing-representative” agent in the model. In Sections IV to VI we illustrate the significance of our findings for the pricing of options in this economy.

II. A Two-period Arrow-Debreu Economy with Heterogeneous Agents

We assume a one-good, two-date exchange economy. The aggregate consumption at date 1 (“tomorrow”) is uncertain. We normalize the scale of consumption so that total consumption at date 0 (“today”), Y_0, equals unity. We denote by Y_n the strictly positive total endowment of future consumption in state n, $n = 1, \ldots N$.

We consider an economy with $H \geq 2$ agents. Each agent i is assumed to have a fixed initial fraction of ownership w_i of the economy’s endowment today and in each state of nature tomorrow. We denote i’s initial-period consumption by y_{i0}, and denote by y_{in} the second-period consumption by agent i in state n. Each of the H agents has time-separable, expected utility preferences that take the form:

$$U_i(y_i) = u_i(y_{i0}) + \beta_i \sum_{n=1}^{N} f_n u_i(y_{in}), \textit{ where } u_i(y) = \frac{y^{1-\gamma_i}}{1-\gamma_i} \tag{1}$$

and where f_n is the strictly positive probability assessment of state n, β_i is agent i’s subjective rate of time discount and γ_i is her constant degree of relative risk aversion. Each agent is thus characterized by the subjective parameters $\{\beta_i, \gamma_i\}$ and the fraction of ownership in the aggregate endowment at dates 0 and 1, w_i; we note that w_i is also individual’s i’s fraction of total wealth. We assume $\beta_i > 0$, $\gamma_i > 0$, and $w_i > 0$.

We assume the existence of a full initial set of Arrow-Debreu markets, so that in equilibrium there are no potential benefits to trade. Let p_n denote the Arrow-Debreu equilibrium price of contingent consumption in state n. Each agent i selects a consumption program which maximizes $U_i(y_i)$ in (1) given her budget constraint,

$$y_{i0} + \sum_{N=1}^{N} p_n y_{in} = w_i \left[Y_0 + \sum_{n=1}^{N} p_n Y_n \right] \tag{2}$$

Let $\tilde{y}_i = \frac{\left[\frac{1-\alpha(Y)}{1-\varpi}\right]^{-\gamma_2}}{Y^{\gamma_2-\gamma_0}}$ be the equilibrium allocation in the economy. As is well-known, given the particular preferences assumed here, there will be no corner solutions and all y_i% will be strictly positive. In equilibrium, all consumers' marginal rates of substitution equal the state prices:

$$p_n = \beta_i f_n \frac{u_i'(y_{in})}{u_n'(y_{i0})} = \beta_i f_n \left[\frac{y_{in}}{y_{i0}} \right]^{-\gamma_i} \textit{for all } i, n \tag{3}$$

The prices p_n are determined in a Walrasian equilibrium so as to equate the demand and supply for all the state-contingent goods.[5]

$$\sum_{i=1}^{H} y_{in} = Y_n, \textit{for all } n \tag{4}$$

Given the particular pattern of tastes in this economy, it is clear that no generality is lost by normalizing $Y_0 = 1$. Let $\omega_i = y_{i0}/Y_0 = y_{i0}$ denote agent i's date 0 share of consumption. The equilibrium prices are, of course, a function of the agents' characteristics $\{\beta_i, \gamma_i, w_i\}$, and of market quantities $\{Y_n\}$. By combining (3) and (4), we can view p_n as determined by:

$$\sum_{i=1}^{H} \omega_i \left[\frac{\beta_i f_n}{p_n} \right]^{1/\gamma_i} = Y_n, \textit{for all } n \tag{5}$$

This equation determines the equilibrium prices p_n as a function of the aggregate consumption quantities Y_n and the agents' taste parameters. However, the prices in (5) are dependent on agents' endogenously-determined shares of initial period consumption $\{\omega_i\}$, instead of their exogenous initial shares of total wealth $\{w_i\}$. This transformation of variables simplifies the presentation below. The equilibrium conditions (3)–(4) and agents' budget constraints (2) establish a one-to-one relation between agents' initial distribution of wealth $\{w_i\}$ and the distribution of initial consumption $\{\omega_i\}$. Given the latter, and given p_n as determined by (5), we can consider the initial wealth fractions as if determined by:

$$w_i = \omega_i \left[\frac{1 + \sum_{n=1}^{N} p_n \left[\beta_i f_n / p_n\right]^{1/\gamma_i}}{1 + \sum_{n=1}^{N} p_n Y_n} \right] \tag{6}$$

III. Identifying Preferences for a Pricing-representative Individual

In this section we identify the characteristics of a "pricing-representative" agent in the above economy. We define a "pricing-representative" agent as one whose tastes are such that if all H agents in the economy had tastes identical to his, then the equilibrium state prices in the economy would remain unchanged. As noted in the previous section, it is not possible to find a "representative consumer" who can mimic market prices for any possible set of aggregate endowments $\{Y_n\}$; we therefore look for preferences which can mimic the prices in the given economy. We assume that the utility function for the pricing-representative agent takes the separable form:

$$U^*(Y) = u_0^*(Y_0) + \beta^* \sum_{n=1}^{N} f_n u^*(Y_n) \tag{7}$$

The form of the utility function of the pricing-representative agent, (7), assumes time-separability and expected utility maximization, but does not impose the additional assumption of a constant elasticity temporal utility function assumed in (1) above for each individual.

What we require from this function is that its marginal rates of substitution be equal to the equilibrium state prices p_n:

$$\beta^* f_n \left[\frac{u^{*'}(Y_n)}{u_0^{*'}(Y_0)} \right] = p_n, \textit{for all } n \tag{8}$$

Equation (8) presents a set of conditions from which we can identify properties of the preferences of the pricing-representative agent. Given our normalization $Y_0 = 1$, we define the probability normalized prices $q(Y)$, by the condition:

$$q(Y) = \beta^* \left[\frac{u^{*'}(Y)}{u_0^{*'}(1)} \right]. \tag{9}$$

The function $q(Y)$ is sufficient to price all state-contingent commodities, since by (8):

$$q(Y_n) = \frac{p_n}{f_n}, \textit{for all } n. \tag{10}$$

We now propose to identify properties of the pricing representing agent by making the additional assumption that the set of states of nature is sufficiently dense, so that every level of positive future aggregate consumption is possible. Given this assumption, it follows by comparing (5) and (8) that we require the pricing function $q(Y)$ to satisfy the implicit condition:

$$\sum_{i=1}^{H} \frac{\omega_i}{Y} \left[\frac{\beta_i}{q(Y)} \right]^{1/\gamma_i} = 1, \; for \; all \; Y > 0 \tag{11}$$

The function $q(Y)$ is implicitly defined by equation (11) for every level of aggregate consumption Y (or in fact for every rate of consumption growth Y/Y_0) by the set of investors' taste parameters, $\{\beta_i, \gamma_i\}$ and by the initial consumption shares $\{\omega_i\}$, which, as shown in equation (6) can be taken as a proxy for the initial endowment shares $\{w_i\}$.

To simplify notation further we make the normalization: $u_0^{*'}(1) = u^{*'}(1) = 1$. Setting $Y = 1$ in condition (9) identifies the time-discount factor β^* of the pricing-representative agent:

$$\beta^* = q(1). \tag{12}$$

By (11) it then follows that

$$\sum_{i=1}^{H} \omega_i \left[\frac{\beta_i}{\beta^*}\right]^{1/\gamma_i} = 1. \tag{13}$$

It can then easily be established that the representative time-discount factor β^* is some average of all agents' time discount factors β_i, and in particular is between $\mathrm{Max}_i\{\beta_i\}$ and $\mathrm{Min}_i\{\beta_i\}$. Conditions (9) and (12) then further identify the temporal utility function $u^*(Y)$ as the solution for the differential equation:

$$u^{*'}(Y) = \frac{q(Y)}{q(1)} \tag{14}$$

Since equation (14) is assumed to hold as an identity for all positive values of Y, we can, by differentiation, define the temporal degree of relative risk aversion of the pricing-representative investor:

$$\gamma^*(Y) = \frac{-Yq'(Y)}{q(Y)}. \tag{15}$$

Proposition 1 *For any Y, $\gamma^*(Y)$ is a harmonic weighted average of individuals' γ_i's. Thus, in particular, $\gamma^*(Y)$ is bounded by $\mathrm{Max}_i\{\gamma_i\}$ from above and by $\mathrm{Min}_i\{\gamma_i\}$ from below.*

Proof: By equation (11), $q(Y)$ is determined as the solution of the implicit condition:

$$F(Y, q) \equiv \sum_{i=1}^{H} \frac{\omega_i}{Y} \left[\frac{\beta_i}{q}\right]^{1/\gamma_i} = 1. \tag{16}$$

It follows that:

$$\frac{\partial F}{\partial Y} = -\sum_i \frac{\omega_i}{Y^2} \left[\frac{\beta_i}{q}\right]^{1/\gamma_i} = -\frac{1}{Y}. \tag{17}$$

and

$$\frac{\partial F}{\partial q} = -\frac{1}{Yq} \sum_i \frac{\omega_i}{\gamma_i} \left[\frac{\beta_i}{q}\right]^{1/\gamma_i} = -\frac{1}{q} \sum_i \frac{\alpha_i}{\gamma_i}. \tag{18}$$

where

$$\alpha_i = \alpha_i(Y) \equiv \frac{\omega_i}{Y} \left[\frac{\beta_i}{q(Y)}\right]^{1/\gamma_i}. \tag{19}$$

This means that:

$$q'(Y) = -\frac{\partial F/\partial Y}{\partial F/\partial q} = -\frac{q(Y)/Y}{\sum_i \alpha_i/\gamma_i}. \tag{20}$$

By its definition in (15),

$$\gamma^*(Y) = \frac{1}{\sum_i \alpha_i/\gamma_i} \tag{21}$$

By equation (3), the weights $a_i(Y)$ that were defined in (19), are simply the second-period consumption shares of agents, when the aggregate endowment is Y. By (11), the α_i sum to one. ■

Proposition 1 shows that the risk aversion of the pricing representative consumer is *not* a simple average of the risk aversions of the individuals in the economy. As we see in the next proposition, this means that for our model, where all the individuals in the economy have constant relative risk aversion, the pricing representative consumer has *decreasing relative risk aversion*.

Proposition 2 *The pricing-representative agent displays decreasing relative risk aversion. The relative risk aversion of the pricing-representative agent will be strictly decreasing if agents differ in their relative risk aversion.*

Proof: By differentiation of (19), and use of (20)–(21):

$$\frac{d\alpha_i}{dY} = \frac{\alpha_i}{Y}\left[\frac{\gamma^*}{\gamma_i} - 1\right] \tag{22}$$

Thus, as Y increases, the weight α_i of those investors with relatively low degree of risk aversion increases. From (21) and (22),

$$\begin{aligned}\frac{d\gamma^*}{dY} &= \frac{-\sum_i\left(\frac{1}{\gamma_i}\right)\left[\frac{d\alpha_i}{dY}\right]}{\left(\sum_i \frac{\alpha_i}{\gamma_i}\right)^2} \\ &= \frac{(\gamma^*)^2}{Y}\left\{\sum_i\left[\frac{\alpha_i}{\gamma_i} - \gamma^*\left(\frac{\alpha_i}{\gamma_i^2}\right)\right]\right\} = \frac{(\gamma^*)^3}{Y}\left\{\left[\sum_i \frac{\alpha_i}{\gamma_i}\right]^2 - \sum_i \frac{\alpha_i}{\gamma_i^2}\right\}\end{aligned} \tag{23}$$

It follows that $d\gamma^*/dY < 0$ if and only if

$$\frac{1}{(\gamma^*)^2} = \left[\sum_i \frac{\alpha_i}{\gamma_i}\right]^2 < \sum_i \frac{\alpha_i}{\gamma_i^2} \tag{24}$$

Looking at the random variable X that obtains the value $1/\gamma_i$ with the probability α_i, we see that the right-hand side above is $E(X^2)$ and the left-hand side is $[E(X)]^2$. Our claim is now established, since the variance of X, $EX^2 - (EX)^2$, has to be positive. ■

As the next proposition shows, in states where the aggregate consumption is very high, the "pricing representative" consumer's RRA looks like the least risk-averse consumer, and *vice-versa*:

Proposition 3

(i) If $Y \to \infty$, then $\gamma^* \to Min_i\{\gamma_i\}$;

(ii) if $Y \to 0$, then $\gamma^* \to Max_i\{\gamma_i\}$.

Proof: (i) Assume that $\gamma_j > \text{Min}_i\{\gamma_i\} = \gamma_k$. By (21), it is sufficient to prove that as $Y \to \infty, \alpha_j \to 0$. Suppose to the contrary that there exists ε_0 and $\overline{Y}$ such that $\alpha_j > \varepsilon_0 > 0$ for $Y > \overline{Y}$. By (21) then, $\gamma^*(Y)$ will be bounded away from γ_k, and there will exist $\varepsilon_1 > 0$ such that $[\gamma^*(Y)/\gamma_k - 1] > \varepsilon_1$ for all $Y > \overline{Y}$. Thus by equation (22) for all $Y > \overline{Y} d\ln(\alpha_k)/d\ln(Y) > \varepsilon_1$. This differential inequality implies that as $Y \to \infty, \alpha_j$ grows to infinity. But $\alpha_j < 1$, so that we have a contradiction. The proof of (ii) is analogous.

■

A numerical example

A numerical example may give some insight into these propositions. Consider a two-date model with 3 equally-likely states at date 1. Aggregate consumption at date 0 is 1, and aggregate date 1 consumption in states 1, 2, and 3 is {0.8, 2, 3}. There are two consumers who have equal initial shares in the consumption at date 0 and in each state of the world at date 1. Each consumer has a utility function with pure time preference $\beta = 0.99$; consumer 1 has RRA $\gamma_1 = 1$ and consumer 2 has RRA $\gamma_2 = 7$.

To solve for the equilibrium consumptions, we find $\{y_{i0}, y_{i1}, y_{i2}, y_{i3}\}$, $i = 1, 2$ such that: a) for $n = 1, 2, 3,$ the state prices for both consumers are equal:

$$\beta f_n \left[\frac{u'(y_{1n})}{u'(y_{10})}\right] = \frac{0.99}{3}\left[\frac{y_{1n}}{y_{10}}\right]^{-1} = \beta f_n \left[\frac{u'(y_{2n})}{u'(y_{20})}\right] = \frac{0.99}{3}\left[\frac{y_{2n}}{y_{20}}\right]^{-7},$$

b) the budget constraints for each consumer are satisfied, and c) the aggregate consumption equals the sum of the individual consumptions on a state-by-state basis (i.e., the market clears).

The reader can confirm that the consumption vectors for consumers 1 and 2 and the resulting state prices in Figure 1 satisfy these 3 conditions. Although both consumers consume more in states where the aggregate consumption is greater, the more risk averse consumer 2 has less variability in her consumption than consumer 1.

One way to calculate the "pricing representative" consumer's RRA, is to solve the following equation for γ_n on a state-by-state basis:

$$\beta f_n \left[\frac{u^{*'}(Y_n)}{u_0^{*'}(Y_0)}\right] = p_n$$

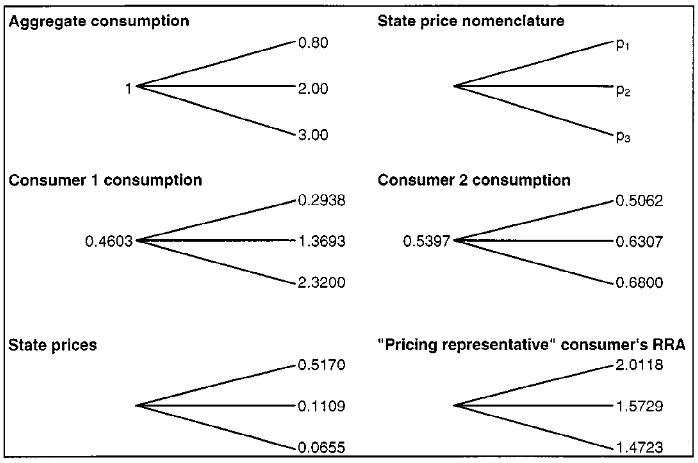

Figure 1. Two consumers with the same time preference and equal endowments divide aggregate consumption of 1 at date 0 and $\{0.8, 2, 3\}$ at date 1 consumption. The coefficient of relative risk aversion of consumer 1 is $\gamma_1 = 1$ and that of consumer 2 is $\gamma_2 = 7$. The figure shows the equilibrium consumption of each consumer, the resulting state prices and the pricing-representative consumer's (decreasing) relative risk aversion.

where $p_n = \{0.5170, 0.1109, 0.0655\}$ are the equilibrium state prices in Figure 1, where $u_0^{*'}(Y_0) = 1$ and where $u^{*'}(Y_n) = Y_n^{-\gamma_n}$. This is a discrete-state analog of the concept of relative risk aversion, as here we have specified the marginal utility of the representative individual at only a finite number of points. The result is that the "pricing representative" consumer's RRA in state 1 (a low consumption state) is higher than the "pricing representative" consumer's RRA in state 3 (a high consumption state). As proved Proposition 2, although both consumers have constant RRA, the "pricing representative" consumers RRA is *decreasing*.

IV. The Pricing of Options on Aggregate Consumption: Preliminaries

In this and the next two sections we apply the results of Section III to the pricing of options in a heterogeneous consumer economy. To simplify the exposition we assume that the economy has only two competitive agents, each endowed with constant relative risk aversion preferences. Each agent is also assumed to believe (correctly) that the probability distribution of aggregate consumption at date 1 is lognormal.

To understand the intuition behind our claim that the pricing of options should be particularly sensitive to heterogeneity among investors, consider the case where the two agents differ in their risk aversion. As shown in the discussion at the end of the previous section, the more risk averse agent seeks to guarantee that the amplitude of her future consumption will be small, and in particular seeks to protect herself against downside risk. As proved formally in Proposition 3, this agent will thus dominate both the date 1 consumption in states of low aggregate consumption and the pricing of contingent consumption in these states. The less risk averse agent, less concerned with protection against downside risk, will correspondingly dominate in the consumption and the pricing of consumption in high states.

Since an out-of-the-money call option (on total consumption, in this framework) offers the upper tail of the distribution, it follows that its price will be influenced primarily by the attitude towards risk of the less risk averse investor. Symmetrically, the pricing of out-of-the-money put options will be particularly influenced by the attitude towards risk of the more risk averse investor. This intuition thus suggests that the pricing of contingent commodities by any "average" agent, with constant relative risk aversion, will underprice the contingent consumption in both tails of the distribution, and also tend to underprice out-of-the-money options.

The intuitive discussion above can be alternatively considered as an illustration of Proposition 2, that an economy with heterogeneous agents with constant relative risk aversion will price assets as if it consisted of a single investor with declining relative risk aversion.

To set the stage for a more formal application of this intuitive logic, we have first to define the relevant assets in this two period economy, and then consider the reference case of asset pricing when the agent are homogeneous. Let $p(Y)$ denote the equilibrium price at date 0 of a unit of date 1 consumption. The interest rate r is determined by the condition that $(1+r)^{-1}$ is the date 0 price of a unit of consumption in every date 1 contingency:

$$(1+r)^{-1} = \int_0^\infty p(Y)dY \tag{25}$$

We identify the future payout of the "market" in this economy as consisting of the entire date 1 endowment. The date 0 market price S, is thus:

$$S = \int_d^\infty Yp(Y)dY \tag{26}$$

The price of a call and a put option on the market with a strike price of X is therefore:

$$C(X) = \int_X^\infty p(Y)(Y - X)dY, \quad P(X) = \int_0^X p(Y)(X - Y)dY \tag{27}$$

Denote by $\alpha = \alpha(Y)$ consumer 1's share of aggregate future consumption when the aggregate future consumption is Y and denote by ω consumer 1's equilibrium share of date 0 consumption. By (3), the consumption share $\alpha = \alpha(Y)$ and the equilibrium price $p = p(Y)$ are jointly determined by the condition:

$$p = \beta_1 f(Y) \left[\frac{\alpha \cdot Y}{\omega}\right]^{\gamma_1} = \beta_2 f(Y) \left[\frac{(1-\alpha) \cdot Y}{1-\omega}\right]^{-\gamma_2}. \tag{28}$$

Given our assumption that both agents believe that the distribution of aggregate future consumption is lognormal:

$$f(Y) = f(Y;\mu,\sigma) = \frac{1}{Y\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma}[\ln Y - \mu]^2\right]. \tag{29}$$

As a reference for the subsequent analysis of the implications of heterogeneity, we now briefly summarize the well-known results for the case where the two agents are identical, so that the identifying index i can be dropped. The proposition below has been proven by Rubinstein (1976), Brennan (1979), and Stapleton and Subrahmanyam (1984):

Proposition 4 *If aggregate date 1 consumption Y is lognormally distributed with mean μ and standard deviation σ, and if all investors share identical time-additive preferences with constant relative risk aversion γ, then:*

(i) The normalized Arrow-Debreu state prices (the "pricing kernel," or the "risk-neutral probabilities"), $(1+r)\,p(Y)$, can be considered as the probability density of a lognormal variable with density $f(Y;\mu-\gamma\sigma^2,\sigma)$.

(ii) The pricing of call options is according to the Black-Scholes formula:

$$C(X) \;=\; BS(X;S,r,\sigma) \equiv SN[d] - \frac{X}{1+r}N[d-\sigma], \tag{30}$$

$$\textit{where } d \;=\; \frac{\ln(s/x)+\ln(1+r)+\sigma^2/2}{\sigma}. \tag{31}$$

V. The Pricing of Options with Heterogeneous Tastes

With the case of homogeneity as a reference point, we now return to the implications of heterogeneity among the two agents in this simple two-period, two-agent economy. Given our assumptions, the preferences of each agent will be represented by two parameters: (β_i, γ_i) for $i = 1, 2$, where all agents believe that Y is lognormally distributed with parameters μ and σ.

To simplify the presentation, we consider here separately the effect of heterogeneity in only one of these parameters at a time.

V.a. The Case of Heterogeneity in Subjective Time Discounting

In this case it is assumed that the only subjective parameter in which agents differ is their discount factor β_i. By solving equation (28) for α, it is clear that in this case each agent's share of second-period consumption will be a constant, independent of aggregate consumption Y. In fact, this economy will generate state prices like the ones in the case of a homogeneous economy, where the representative agent has a discount factor β^* defined by:

$$(\beta^*)^{1/\gamma} = \omega\beta_1^{1/\gamma} + (1-\omega)\beta_2^{1/\gamma} \tag{32}$$

As a result, Proposition 4 will apply, and a Black-Scholes formula will price call and put options.

V.b. The Case of Heterogeneity in Risk Aversion

This is the main case for which the propositions of Section III should apply. Given aggregate consumption Y, let $p(Y) = p(Y; \gamma_1, \gamma_2)$ denote the state price, $C(X; \gamma_1, \gamma_2)$ the price of a call option with strike X, and $P(X; \gamma_1, \gamma_2)$ the price of a put option with strike X. The consumption share of the first agent, $\alpha(Y) = \alpha(Y; \gamma_1, \gamma_2)$ is implicitly determined by the condition:

$$\left(\frac{\alpha(Y)Y}{\omega}\right)^{-\gamma_1} = \left(\frac{(1-\alpha(Y))Y}{1-\omega}\right)^{-\gamma_2} \tag{33}$$

There is no analytic solution for the function $\alpha(Y)$ in this case. However, the following proposition can be obtained as a corollary of Propositions 1–3:

Proposition 5 *If the two agents differ only in their coefficient of relative risk aversion and if $\gamma_1 < \gamma_2$, then $\alpha(Y)$, the future consumption share of the less risk averse first agent, will be monotonically increasing in Y, with* $\lim_{Y\to 0} \alpha(Y) = 0,\ \lim_{Y\to\infty} \alpha(Y) = 1$.

Proof: By Proposition 1, $\gamma^*(Y) > \gamma_1$. Equation (22) thus establishes monotonicity, and Proposition 3 shows that the limits are as stated. ■

As was proved in Proposition 2, as a result of the endogenous non-constant sharing of consumption, the pricing function $p(Y)$ in the case of heterogeneous agents can be interpreted as displaying declining relative risk aversion. It follows from this implication that the pricing kernel $(1 + r)p(Y)$ will no longer be lognormally distributed, as in the homogeneous case, and the Black-Scholes formula will no longer apply. The relevant issue, however, is to identify what specifically will distinguish option pricing in this heterogeneous economy from the option pricing that would apply in a "similar" homogeneous economy. For this purposes we seek to compare the case of the heterogeneous economy with another similar, yet homogeneous, economy where both agents share some "average" constant coefficient of risk aversion, γ^0. No matter how this average γ^0 is chosen, the following proposition applies.

Proposition 6 *For any γ^0, such that $\gamma_1 < \gamma^0 < \gamma_2$, there are two positive values Y_{high} and Y_{low} so that state prices $p(Y; \gamma_1, \gamma_2) > p(Y; \gamma^0, \gamma^0)$ if either $Y > Y_{high}$ or $0 < Y < Y_{low}$. As a result,*

(i) For sufficiently high X, $C(X; \gamma_1, \gamma_2) > C(X; \gamma^0, \gamma^0)$,

(ii) For X sufficiently close to zero, $P(X; \gamma_1, \gamma_2) > P(X; \gamma^0, \gamma^0)$.

Proof: From Proposition 5 it follows that since $\alpha(Y; \gamma_1, \gamma_2)$ increases monotonically towards 1, and since $\gamma_0 > \gamma_1$, when Y approaches infinity, the following price ratio approaches infinity:

$$\frac{p(Y; \gamma_1, \gamma_2)}{p(Y; \gamma_0, \gamma_0)} = \frac{\beta\left[\frac{\alpha(Y)Y}{\omega}\right]^{-\gamma_1} f(Y)}{\beta Y^{-\gamma_0} f(Y)} = \left[\frac{\alpha(Y)}{\omega}\right]^{-\gamma_1} Y^{\gamma_0-\gamma_1}. \tag{34}$$

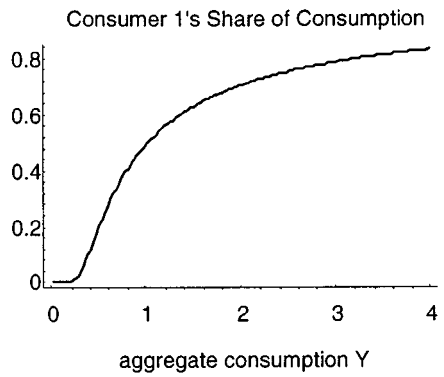

Figure 2. In an economy with two equally-endowed consumers, consumer 1 has relative risk aversion $\gamma_1 = 1$ and consumer 2 has relative risk aversion $\gamma_2 = 7$. Given aggregate consumption distributed lognormally with $\mu = 0.15$, $\sigma = 0.3$, we calculate the equilibrium consumption share of consumer 1. As suggested by Proposition 5, this function is monotonically increasing.

Similarly, since $\gamma_0 < \gamma_2$, it follows that when Y approaches zero, the following price ratio approaches infinity:

$$\frac{P(Y;\gamma_1,\gamma_2)}{P(Y;\gamma_0,\gamma_0)} = \frac{\beta\left[\frac{(1-\alpha(Y))Y}{1-\omega}\right]^{-\gamma_2} f(Y)}{\beta Y^{-\gamma_0} f(Y)} = \frac{\left[\frac{1-\alpha(Y)}{1-\omega}\right]^{-\gamma_2}}{Y^{\gamma_2-\gamma_0}}. \tag{35}$$

This proves the first part of the proposition and the implications concerning the pricing of far out-of-the-money put and call options now follow. ■

A Calibrated Example

Since our concern is to examine the impact of heterogeneity of risk aversion on the pricing of options and since there is no closed-form solution for prices in this case, we illustrate the potential magnitude of the impact of Proposition 5 and 6, by use of a calibration example. In the example below we set $\beta = 0.9$, $\gamma_1 = 1$, $\gamma_2 = 7$, $\mu = 0.15$, $\sigma = 0.3$. We set the initial consumption shares of the agents so that in equilibrium they have equal endowments. For these parameter values, the date 0 market value of future aggregate consumption is 0.8651 and the market interest rate is 19.07%.

Figure 2 presents the shape of the consumption share function $a(Y)$ for this calibration. As suggested by Proposition 5, this function is indeed monotonically increasing.

Define $C(X;\gamma_1,\gamma_2)$ to be the price of a call option with exercise price X in an economy in which the two investors have risk aversions γ_1 and γ_2:

$$C(X;\gamma_1,\gamma_2) = \int_X^\infty p(Y;\gamma_1,\gamma_2)(Y - X)dY.$$

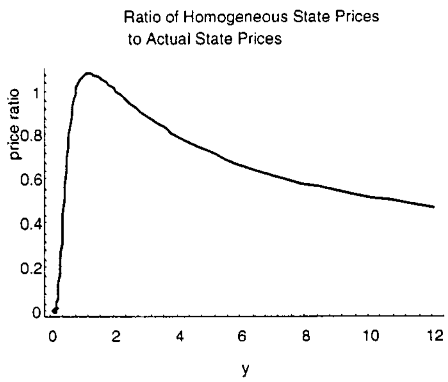

Figure 3. In an economy with two equally-endowed consumers, consumer 1 has relative risk aversion $\gamma_1 = 1$ and consumer 2 has relative risk aversion $\gamma_2 = 7$. Figure 3 compares the state prices in this economy with those of a homogeneous agent economy where all parameters are identical and where both investors share the same "average" degree of risk aversion: $\gamma_1 = \gamma_2 = \gamma^0$, defined so that the homogeneous economy will have the correct price for an at-the-money call option.

In the calibrations below, we compare the heterogeneous-economy call option price $C(X; \gamma_1, \gamma_2)$, where $\gamma_1 = 1, \gamma_2 = 7$, with the call option price $C(X; \gamma^0, \gamma^0)$ in a homogeneous agent economy where all the parameters are identical and where both investors share the same "average" degree of risk aversion: $\gamma_1 = \gamma_2 = \gamma^0$. It is not obvious how to define this "average" γ^0. At this point we choose to define it so that the homogeneous economy will have the correct price for an at-the-money call option: $X = 0.8651$.[6] That is, γ^0 was defined to solve $C(0.8651; \gamma_1, \gamma_2) = C(0.8651; \gamma^0, \gamma^0)$.

Figure 3 compares the prices of this "average" consumer (for whom we numerically obtained that $\gamma^0 = 1.58$) with the actual prices in the economy. In the graph we show the ratio of these prices $p(Y; \gamma^0, \gamma^0)/p(Y; \gamma_1, \gamma_2)$. As proved in Proposition 6, the actual state prices exceed the prices in any homogeneous consumer economy for both consumption tails.

In accordance with Propositions 3 and 6, the "average" consumer economy generates lower state prices than the actual heterogeneous economy, both for low and for very high levels of aggregate consumption. As a direct result of this key finding, it is not surprising that the homogeneous "average" economy will tend to underprice out-of-the-money calls.[7] This result is depicted in Figure 4 below, which shows the ratio of the actual call option price in the heterogeneous consumer economy $C(X; 1, 7)$ and the Black-Scholes price $C(X; \gamma^0, \gamma^0)$, for that homogeneous economy with the "average" constant relative risk aversion γ^0 as defined above.

The main finding displayed in Figure 4 is that options that are away from the money (whether in or out of the money) are *more expensive* in this heterogeneous consumer economy than in the Black-Scholes case. The ratio of the call prices depicted in Figure 4 depends, of course, on how we determine the "average" γ^0. Were we to choose to normalize on a Black-Scholes option with a different strike price, we would determine a different "average" γ^0. We return to this topic in Section V.d. below. However, as proved in Proposition

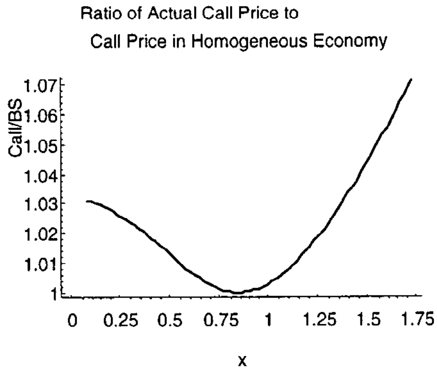

Figure 4. In an economy with two equally-endowed consumers, consumer 1 has relative risk aversion $\gamma_1 = 1$ and consumer 2 has relative risk aversion $\gamma_2 = 7$. Figure 4 shows the ratio of the actual call option price in the heterogeneous consumer economy $C(X; 1, 7)$ and the Black-Scholes price $C(X; \gamma^0, \gamma^0)$, for that homogeneous economy with the "average" constant relative risk aversion γ^0 as defined in Figure 3.

6, the pattern that emerges for out-of-the money options is robust to the selection of the "average" γ^0.

V.c. Implied Volatility: Smiles and Heterogeneous Consumers

Given the difficulties in estimating the volatility σ, the Black-Scholes formula is often presented empirically in terms of the stock volatility that is implied by the market pricing of call options with alternative strike prices. Consistency with the Black-Scholes formula should imply a horizontal curve for the implied volatility as a function of the strike price, but the empirical pattern that researchers typically find displays a "smile."

Given the market interest rate, $r(\gamma_1, \gamma_2)$, and the stock value $S(\gamma_1, \gamma_2)$ in the heterogeneous economy we now solve the Black-Scholes formula in (30) for the implied volatility. That is, for the function BS(.) in (30) and for each X, we determine $\sigma = \sigma(X)$, such that when $\gamma_1 = 1, \gamma_2 = 7$, the following identity holds: $BS(X; S(\gamma_1, \gamma_2), r(\gamma_1, \gamma_2), \sigma) = C(X; \gamma_1, \gamma_2)$.

Figure 5 shows the implied volatility of the call option in our calibrated example (where $\sigma = 0.3$ is the true volatility used in the lognormal density function). A smile pattern is evident: The implied volatility for out-of-the money options is lower than that for in-the-money options. This is the pattern that is presented (among others) by Rubinstein (1994). As the exercise price of the options gets large, the implied volatility in this simulated example approaches the actual, 30%, volatility of the underlying consumption process from above. This means that for this particular case the implied volatility is *everywhere larger* than the volatility of the underlying consumption process.[8]

V.d. Alternative Normalizations

In the example of section V.b., we chose a "representative" consumer by determining a risk aversion coefficient γ^0 so that the price of an at-the-money call in the homogeneous

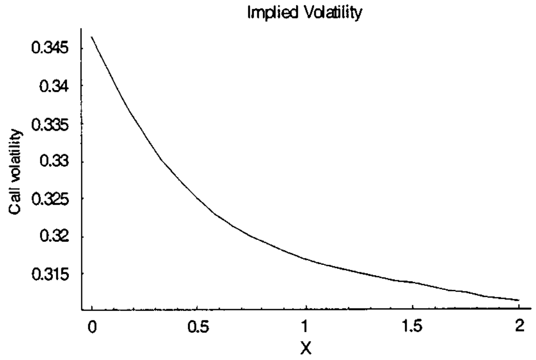

Figure 5. Figure 5 shows the implied volatility of the call option in our calibrated example (where $\sigma = 0.3$ is the true volatility used in the lognormal density function). The implied volatility for out-of-the money options is lower than that for in-the-money options. For this particular case the implied volatility is *everywhere larger* than the volatility of the underlying consumption process.

consumer economy equals that of an at-the-money call in the heterogeneous consumer economy—$C(S(\gamma_1, \gamma_2); \gamma_1, \gamma_2) = C(S(\gamma_1, \gamma_2); \gamma^0, \gamma^0)$. There are several ways to choose such a normalization:

Case 1. We shall refer to the normalization of section V.b. as Case 1:

$$C(S(\gamma_1, \gamma_2); \gamma_1, \gamma_2) = C(S(\gamma_1, \gamma_2); \gamma^0, \gamma^0).$$

As can be see from Table 1 below, for our base values of $\gamma_1 = 1$ and $\gamma_2 = 7$, this means that $\gamma_0 = 1.58$ and $S(\gamma_1, \gamma_2) = 0.8371$ so that

$$\begin{aligned} 0.2004 &= C(S(\gamma_1, \gamma_2); \gamma_1, \gamma_2) = C(0.8371; 1, 7) \\ &= C(0.8371; 1.58, 1.58) = C(S(\gamma_1, \gamma_2); \gamma^0, \gamma^0) \end{aligned}$$

Case 2. Instead of normalizing on an at-the-money call in the heterogeneous economy, we could normalize on an at-the-money call in each economy. That is, we choose γ^0 so that $C(S(\gamma_1, \gamma_2); \gamma_1, \gamma_2) = C(S(\gamma^0, \gamma^0); \gamma^0, \gamma^0)$. As can be seen from Table 1 below, this means that:

$$\begin{aligned} 0.1844 &= C(S(\gamma_1, \gamma_2); \gamma_1, \gamma_2) = C(0.8371; 1, 7) \\ &= C(0.8003; 2.26, 2.26) = C(S(\gamma^0, \gamma^0); \gamma^0, \gamma^0) \end{aligned}$$

Case 3. In this case we find the "average" relative risk aversion γ^0 which matches the riskless interest rates in both economies:

$$\int_Y p(Y, \gamma_1, \gamma_2) dY = \int_Y p(Y, \gamma^0, \gamma^0) dY$$

Table 1. Comparing four normalizations. Different methods of finding the “average” relative risk aversion γ^0.

	Base Case: actual heterogeneous economy	**Case 1:** normalizing on at-money-call in actual economy	**Case 2:** normalizing on at-money-call in “average” economy	**Case 3:** normalizing on interest rates	**Case 4:** normalizing on market values
“average” γ^0		1.58	2.26	2.78	1.29
market interest rate	19.07%	25.87%	23.97%	19.07%	25.10%
market value	0.8651	0.8371	0.8003	0.7968	0.8651
At-the-money Call ($x = S$)	0.1844	0.2004	0.1844	0.1649	0.2038
Call ($x = 0.5S$)	0.5020	0.5046	0.4776	0.4610	0.5194
Call ($x = 1.5^*S$)	0.0409	0.0472	0.0419	0.0338	0.0473

Notes: a. The calibrations assume a lognormal aggregate consumption process with $\mu = 15\%$, $\sigma = 30\%$. The original economy has two consumers with equal wealth shares and relative risk aversions $\gamma_1 = 1$ and $\gamma_2 = 7$; each consumer has a pure time discount factor $\beta = 0.9$.

b. In cases 2, 3, 4 there is one item in the column which matches a corresponding item for the base case. The exception is case 1, in which we determine the “average gamma” γ_0 by solving $C(S(\gamma_1, \gamma_2); \gamma_1, \gamma_2) = C(S(\gamma_1, \gamma_2); \gamma^0, \gamma^0)$. In case 1 the option price for an at-the-money option is calculated by $C(S(\gamma^0, \gamma^0); \gamma^0, \gamma^0)$.

In Table 1 below it can be seen that for this case:

$$\begin{aligned} 19.07\% &= \int_Y p(Y, \gamma_1, \gamma_2)\, dY = \int_Y p(Y, 1, 7)\, dY \\ &= \int_Y p(Y, 2.78, 2.78)\, dY = \int_Y p(Y, \gamma^0, \gamma^0)\, dy \end{aligned}$$

Case 4. In this case we find the “average” relative risk aversion γ^0 which matches the market values in both economies:

$$\begin{aligned} S(1, 7) = 0.8651 &= \int_Y p(Y, \gamma_1, \gamma_2) Y\, dY = \int_Y p(Y, 1, 7) Y\, dY \\ &= \int_U p(Y, \gamma^0, \gamma^0) Y\, dY = S(1.29, 1.29) \end{aligned}$$

Table 1 summarizes some relevant results for these four cases, and Figure 6 shows the ratios of the actual market price to the homogeneous consumer market price (i.e., the Black-Scholes price) for a range of exercise prices.

It is clear from Table 1 and Figure 6 that the asset prices and call price ratios of the actual and homogeneous-equivalent economies are very sensitive to the selected form of

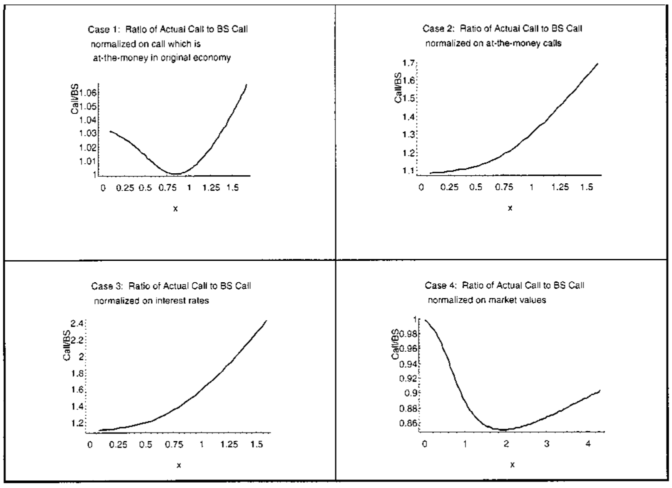

Figure 6. Four Different Normalizations. For each normalization, we show the ratio of the actual call prices to the call price in the "equivalent" homogeneous-consumer economy (this latter price is the Black-Scholes price).

normalization. Depending on the normalization, some option prices may be found to be "underpriced" relative to the Black-Scholes case, and others to be "over-priced." Still, as proved in Proposition 6, *ultimately* (that is, for a high enough strike price), the prices of the calls in our heterogeneous consumer economy will be larger than the Black-Scholes price in any "equivalent" homogeneous economy.

VII. Conclusion

People are different. Some are bold and daring, while others are overcautious. Such diversity is indeed one of the main economic rationales for Pareto-improving trade, and has been particularly emphasized in relation to speculative markets. In this paper we consider equilibrium option pricing in a simple two-period economy that is characterized by heterogeneity among agents. We demonstrate that an economy in which agents have *constant* yet heterogeneous degrees of relative risk aversion will price assets as though it had a single "pricing representative" agent who displays *decreasing* relative risk aversion. This result was shown to imply that the pricing kernel has fat tails and yields option prices which do not conform to the standard Black-Scholes formula. Solving for the implied volatility of either call or put options results in this case in a "smile"pattern, typical of those derived in practice.

Our explanation of heterogeneity as the source for this empirically observed phenomenon is simple and intuitive. It seems to fit Rubinstein's (1994) interpretation of the "over-pricing"of out-of-the-money put options on the S&P 500 index as an indication of "crash-o-phobia." Rubinstein's term suggests that those who seek to hold out-of-the-money put options as protection against crashes are characterized by relatively high risk aversion. If one were to assume that all investors share the same attitude towards risk and probability beliefs with regard to market crashes, there would be no explanation why some investors hold these extreme put options, while others write them. In addition, the very complexity of the implied binomial tree that Rubinstein derives suggests to us that it is likely to be the equilibrium outcome of an interaction among diverse investors, rather than to reflect uniform attitudes towards risk shared unanimously by all investors. While it is convenient to portray the economy through the construct of a fictitious "representative" investor, this convenience should not blind us to ignore the serious aggregation problems that are involved by such a construct, or to regard as innocuous the practice of endowing the fictitious "representative" investor with preferences and probability beliefs that may be "reasonable" only for the actual investors in the economy.

Acknowlegements

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Notes

1. For a recent criticism of the practice of postulating a single "representative" agent see Kirman (1992). A number of recent studies have considered the case of heterogeneity of a different kind: agents who are ex-ante identical end up heterogeneous ex-post, due to the existence of idiosyncratic endowment shocks and due to market imperfections that impede insurance against such shocks (see Mankiw (1986) and the many studies surveyed by Heaton and Lucas (1995)). We should further note that there are articles within the representative-agent framework where the preferences of the representative agent are not of the constant elasticity type.
2. As summarized by Hirshleifer and Riley (1979), with regard to futures trading: "Among the possible determinants of speculative activity, John Maynard Keynes and John Hicks. . . have emphasized differential *risk aversion* . . . . In contrast to these views, Holbrook Working has denied that there is any systematic difference as to risk-tolerance between those conventionally called speculators and hedgers. Working emphasizes, instead, differences of *beliefs* (optimism or pessimism) as motivating futures trading."
3. This point was forcefully made by Leland (1980) in his discussion of portfolio insurance.
4. Constantinides's representative agent is "representative" only for a given set of endowments and will not price assets correctly if there is a change in the stochastic endowment. We thus identify the Constantinides representative agent as "pricing-representative." As Rubinstein (1974) has shown, conditions under which there exists a consumer who is universally representative are extremely restrictive (see also the survey by Shafer and Sonnenschein (1982)).
5. Market clearing for first-period consumption is guaranteed by Walras's Law.
6. In Section V.d. below we explore the implications of alternative methods for selecting γ^0.
7. Franke, Stapleton, and Subrahmanyam (2000) show that the change in sign exhibited by this difference is a necessary condition for a volatility "smile" when the representative consumer has decreasing relative risk aversion.

8. Franke, Stapleton, and Subrahmanyam (2000) interpret this as meaning that options are "too expensive," in the sense that their price will always be greater than the Black-Scholes price. Mathur and Ritchken (1999) have a similar conclusion. As we show in the section V.d., this conclusion is sensitive to the normalization of the option price.

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